Geodesic deviation and gravitational waves

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The detection of gravitational waves based on the geodesic deviation equation is discussed. In particular, it is shown that the only non-vanishing components of the wave field in the conventional traceless-transverse gauge in linearized general relativity do not enter the geodesic deviation equation, and therefore, apparently, no effect is predicted by that equation in that specific gauge. The reason is traced back to the fact that the geodesic deviation equation is written in terms of a coordinate distance, which is not a directly measurable quantity. On the other hand, in the proper Lorentz frame of the detector, the conventional result described in standard textbooks holds.

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I. GEODESIC DEVIATION

We begin by giving a short derivation of the equation of geodesic deviation, without reference to a specific theory of gravitation. It is only assumed that a free falling particle follows a geodesic with respect to a certain, unspecified, connection.

Consider a particle with wordline $x^i(\tau)$ following a geodesic

$$\mathrm{D}u^i = \frac{\mathrm{d}u^i}{\mathrm{d}\tau} + \Gamma^i_{kl}(x)u^k u^l = 0, \tag{1}$$

where $u^i = \mathrm{d}x^i/\mathrm{d}\tau$. Obviously, only the symmetric part of the connection enters the geodesic equation, and we will therefore assume a symmetric connection from now on. Metric compatibility (or even the existence of a metric) is not necessary for our arguments.

Next, consider a neighboring geodesic $\tilde{x}^i(\tau) = x^i(\tau) + \xi^i(\tau)$, following

$$\mathrm{D}\tilde{u}^i = \frac{\mathrm{d}\tilde{u}^i}{\mathrm{d}\tau} + \Gamma^i_{kl}(\tilde{x})\tilde{u}^k\tilde{u}^l = 0. \tag{2}$$

From $\tilde{x}^i - x^i = \xi^i$, we find (the dot denoting the derivative with respect to the curve parameter τ)

$$\begin{aligned}\ddot{\xi}^i &= \dot{\tilde{u}}^i - \dot{u}^i \\ &= -\Gamma^i_{kl}(\tilde{x})\tilde{u}^k\tilde{u}^l - \Gamma^i_{kl}(x)u^k u^l \\ &= -(\Gamma^i_{kl} + \Gamma^i_{kl,m}\xi^m)(u^k + \dot{\xi}^k)(u^l + \dot{\xi}^l) - \Gamma^i_{kl}u^k u^l,\end{aligned}$$

where we have expanded Γ^i_{kl} around x^i to first order in ξ^i. To the same order, we finally find

$$\ddot{\xi}^i = -\Gamma^i_{lm,k}u^l u^m \xi^k - 2\Gamma^i_{kl}u^k\dot{\xi}^l. \tag{3}$$

This is the geodesic deviation equation, although still in a non-explicitly covariant form. We now introduce covariant derivatives

$$\mathrm{D}\xi^i = \dot{\xi}^i + \Gamma^i_{kl}\xi^k u^l, \tag{4}$$

and

$$\begin{aligned}\mathrm{D}^2\xi^i &= (\dot{\xi}^i + \Gamma^i_{kl}\xi^k u^l)^{\cdot} + \Gamma^i_{mj}(\dot{\xi}^m + \Gamma^m_{kl}\xi^k u^l)u^j \\ &= \ddot{\xi}^i + \Gamma^i_{kl,m}\xi^k u^l u^m + \Gamma^i_{kl}\dot{\xi}^k u^l + \Gamma^i_{kl}\xi^k[-\Gamma^l_{pj}u^p u^j] \\ &\quad + \Gamma^i_{mj}\dot{\xi}^m u^j + \Gamma^i_{mj}\Gamma^m_{kl}\xi^k u^l u^j,\end{aligned}$$

where the term in $\dot{u}^l$ has been eliminated with the geodesic equation. Inserting the expression for $\ddot{\xi}^i$, eq. (3), we find

$$\begin{aligned}\mathrm{D}^2\xi^i &= \left(\Gamma^i_{lk,m} - \Gamma^i_{lm,k} + \Gamma^i_{jm}\Gamma^j_{lk} - \Gamma^i_{jk}\Gamma^j_{lm}\right)u^m u^l \xi^k \\ &= R^i{}_{lmk}u^m u^l \xi^k.\end{aligned} \tag{5}$$

Both equations (3) and (5) are completely equivalent and can be used according to convenience.

Let us now consider a coordinate system where $\Gamma^i_{kl} = 0$ on the geodesic $x^i(\tau)$ under investigation (*proper Lorentz frame*). For Γ^i_{kl} to vanish along the complete geodesic, we must have $(\Gamma^i_{kl})^{\cdot} = 0$. Then, obviously, we have $\mathrm{D}^2\xi = \ddot{\xi}^i$, and (5) reduces to

$$\ddot{\xi}^i = \left(\Gamma^i_{lk,m} - \Gamma^i_{lm,k}\right)u^m u^l \xi^k = R^i{}_{lmk}u^m u^l \xi^k, \tag{6}$$

a relation that can be found in many textbooks. There is nothing wrong with this equation, but it can easily lead to misconceptions. Indeed, in the same coordinate system, we find from (3) the equation

$$\ddot{\xi}^i = -\Gamma^i_{lm,k}u^l u^m \xi^k, \tag{7}$$

which differs from (6) by the term $\Gamma^i_{lk,m}u^m u^l \xi^k$. There is still no contradiction, since $\Gamma^i_{lk,m}u^m = (\Gamma^i_{lk})^{\cdot} = 0$, meaning that the first term in (6) is zero anyway.

We also observe that there is actually no need to assume that Γ^i_{kl} is zero along the complete geodesic (as long as we are interested only in one point), because the term containing $(\Gamma^i_{kl})^{\cdot}$ drops out anyway from eq. (5), as is obvious in the form (3). The simple requirement $\Gamma^i_{kl} = 0$ at the point under investigation is sufficient to obtain the result (7). Note, however, that for $(\Gamma^i_{kl})^{\cdot} \neq 0$, the form (6) is not correct, since a further term arises from $\mathrm{D}^2\xi^i$.

II. GRAVITATIONAL WAVES

The previous considerations were quite general. In this section, we assume that Γ^i_{kl} is the Christoffel connection of the metric g_{ik} and the dynamics are governed by the field equations of general relativity.

Consider now a small perturbation h_{ik} on a flat background, i.e., $g_{ik} = \eta_{ik} + h_{ik}$. In the linear approximation, general relativity leads to wave fields that upon imposing appropriate gauge conditions, can be assumed to satisfy the so-called traceless-transverse gauge conditions, i.e., $h_{i0} = 0$, $h^{\alpha\beta}{}_{,\beta} = 0$, $h^{\alpha}{}_{\alpha} = 0$ (where $i, k, \cdots = 0, 1, 2, 3$, and $\alpha, \beta, \cdots = 1, 2, 3$). For the rest, $h_{\alpha\beta}$ is a solution to the wave equation in the flat Minkowski background. For details on the linearized theory, the gauge freedom and the wave solutions, we refer to the standard textbooks [1, 2, 3, 4].

Let us consider the case where initially, we have $u^i = (u^0, 0, 0, 0)$. The geodesic equation (to first order in h_{ik}) then reduces to $\dot{u}^i = \Gamma^i_{00} u^0 u^0 = (h^i{}_{0,0} - \frac{1}{2} h_{00}{}^{,i}) u^0 u^0$. For the traceless-transverse wave, this is obviously zero, meaning that a particle initially at rest will remain at rest for any time. This is a specific feature of the coordinate system related to the above gauge choice.

Let us now proceed in the way of standard textbooks. We start from (6). For $u^\alpha = 0$ (initially) and $\xi^0 = 0$ (as a result of an appropriate choice of the curve parameters of the neighboring geodesics), we find, to first order in h_{ik}

$$\ddot{\xi}^\alpha = \frac{1}{2} \ddot{h}^\alpha{}_\beta \xi^\beta, \tag{8}$$

where we have used the fact that $u^0 = 1 + \mathcal{O}(h)$ and $h^{\alpha\beta}{}_{,0,0} = \ddot{h}^{\alpha\beta} + \mathcal{O}(h^2)$ (in other words, to the required order, we can identify the curve parameter with the time coordinate).

This is the result conventionally presented in literature. Similar relations have already been used by Weber in the context of resonant-mass wave detectors (see, e.g., eqs. (14) and (15) of [5]) and equation (8) can be found in most textbooks on general relativity, see for instance [1], [2] and [3]. The right hand side of (8) describes the force acting on two neighboring particles hold at fixed (coordinate) positions.

At first sight, one might argue that eq. (8) is incorrect for the following reason: Equation (6) was derived under the assumption $(\Gamma^i_{kl})\dot{} = 0$, meaning that the first term in (6) is zero. But it is exactly from this term that the right hand side of (8) emerges. Thus, if we insist in using eq. (6), we have to conclude that $\ddot{h}^{\alpha\beta} = 0$. This is not the case for a wave solution, however. The other way around, if we insist that $\ddot{h}^{\alpha\beta} \neq 0$ (as should be the case for a wave solution), then obviously $(\Gamma^i_{kl})\dot{} \neq 0$, and thus, we cannot use the form (6) of the geodesic deviation equation. In summary, eq. (6) is not consistent with the traceless-transverse gauge condition.

On the other hand, if we use eq. (7) instead, we simply find, with the same conditions used above ($u^\alpha = 0, u^0 = 1 + \mathcal{O}(h), \xi^0 = 0$) that we have $\ddot{\xi}^\alpha = 0$, meaning that no force is applied by the wave on the particles at rest.

Is this the correct result? Well, (7) was derived from (3) assuming that $\Gamma^i_{kl} = 0$ at the (spacetime) point in question, which is less stronger than its vanishing on the complete geodesic, but is still not suitable for the present application. To find the correct relation, it is secure to start directly from (3), which holds in full generality. The result is

$$\ddot{\xi}^\alpha = -\dot{h}^\alpha{}_\beta \dot{\xi}^\beta. \tag{9}$$

In particular, for particles initially at rest (i.e., at fixed coordinate positions), we also have $\dot{\xi}^\alpha = 0$, and thus

$$\ddot{\xi}^\alpha = 0. \tag{10}$$

This equation seems to be in contradiction with equation (8), which is claimed to hold in standard textbooks. Note that eq. (10) also insures that $\dot{\xi}^\alpha$ remains zero at all times, and thus, the equation remains valid to all times.

III. DISCUSSION

In the first version of this paper, we concluded that the standard literature result (8) is incorrect and that the correct relation is given by (10). In the meanwhile, it was pointed out to us by experts on gravitational waves that this conclusion is not entirely correct. Infact, we have misinterpreted the standard literature result, and the truth is that both (8) and (10) are correct, when interpreted in the right way.

Let us begin with the relation (10). It was derived directly from (3), using the traceless transverse gauge for h_{ik} and assuming the usual conditions ($u^\alpha = 0$, $\xi^0 = 0$). It is therefore true that in the traceless transverse gauge, no coordinate acceleration is induced by the passing wave between particles at rest. As we have pointed out already in the first version, this does not mean that there is no physical effect of the wave on the detector, because even if the coordinate distance between particles does not change, the proper distance between them will change, and this is physically measurable. So far, our conclusions of the first version were correct.

From the apparent discrepancy between (8) and (10), we then concluded that the standard result (8) must be wrong. This, however, is not true. Both equations simply refer to different reference frames. Indeed, as we have shown, eq. (6) holds in the proper Lorentz reference frame of the detector. Thus, in order to apply that equation to gravitational waves, we should describe the perturbation h_{ik} in the same reference frame. However, the only way h_{ik} enters (6) is via $R^i{}_{lmk}$, which is gauge invariant. Thus, in order to evaluate $R^i{}_{lmk}$, we can use a gauge of our choice, the result will always be the same. The result is of course eq. (8).

The important thing here is that, although we have used the traceless transverse gauge to evaluate $R^i{}_{klm}$, the final relation (8) is valid only in the proper Lorentz reference frame of the detector. It is therefore not in contradiction with the result (10), which holds only in the traceless transverse gauge. The confusion originates from the fact that in eq. (8), we have expressed $R^i{}_{klm}$ with the help of h_{ik} in the traceless transverse gauge, which is not the metric perturbation in the reference frame the equation actually holds. In other words, in the proper Lorentz reference frame, the metric in not given by $g_{ik} = \eta_{ik} + h_{ik}$.

This might appear trivial a posteriori, but judging from several referee reports of well established journals, which contained various confusing arguments against publication of the first version of the paper, but which did not uncover the actual reason for our erroneous conclusions, it seems that we are not the only ones to which the situation was not entirely clear. For that reason, we wrote this second version of the paper rather than to simply withdraw the original preprint. Further, an article has been pointed out to us [6] which seems to contain conclusions similar to those of our original paper, namely that there is no effect of a gravitational wave on neighboring particles if they are initially at rest. Again, this conclusion should be related to the use of a specific reference frame.

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